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Simplified boson realization of the $so_q(3)$ subalgebra of $u_q(3)$ and matrix elements of $so_q(3)$ quadrupole operators

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Abstract. A simplified boson realization of the $so_q(3)$ subalgebra of $u_q(3)$ is constructed. A simplified form of the corresponding $so_q(3)$ basis states is obtained. The reduced matrix elements of a special second-rank tensor operator (quadrupole operator) are calculated in the $so_q(3)$ basis.

1. Introduction

The construction of chains of subalgebras of a given q-algebra is a non-trivial problem, since the existence of a chain of subalgebras of the corresponding Lie algebra does not guarantee the existence of the q-analogue of this chain. In particular, the $so_q(3)$ subalgebra of $u_q(3)$ has attracted much attention [1–10], since its classical analogue is a basic ingredient of several nuclear models, as the Elliott model [11], the $su(3)$ limit of the interacting boson model (IBM) [12] and the interacting vector boson model (IVBM) [13]. The aim of the present paper is to compute the matrix elements of the $so_q(3)$ quadrupole operator in the $u_q(3) \supset so_q(3)$ basis (for the most symmetric $u_q(3)$ representation). To this purpose we use the results obtained in [1, 6].

In section 2 we introduce a set of modified operators, in terms of which the elements of $so_q(3)$ algebra, i.e. the operators of the q-deformed angular momentum, are expressed in a relatively simple form. In section 3 we express the basis of the q-deformed $so_q(3) \subset u_q(3)$ for the case of the most symmetric representation $[\lambda, 0, 0]$ of $u_q(3)$. In section 4 we also construct $so_q(3)$ vector operators and in section 5 the reduced matrix elements of a special second-rank tensor operator (quadrupole operator) are calculated in the $so_q(3)$ basis.

2. Simplified form of the $so_q(3)$ subalgebra of $u_q(3)$

In this paper we follow the approach of [1, 2], in which a boson realization of the $so_q(3)$ subalgebra of $u_q(3)$ in terms of q-deformed bosons [14, 15] is constructed. The three independent q-deformed boson operators b_i and $b_i^\dagger$ ($i = +, 0, -$) satisfy the commutation relations

$$[N_i, b_i^\dagger] = b_i^\dagger \qquad [N_i, b_i] = -b_i \qquad b_i b_i^\dagger - q^{\pm 1} b_i^\dagger b_i = q^{\mp N_i} \tag{1}$$

where N_i are the corresponding number operators.

It was shown [1] that in the Fock space of the totally symmetric representations $[N, 0, 0]$ of $u_q(3)$ the angular momentum operators i.e. the elements of $so_q(3)$ algebra, have the form

$$\begin{aligned} L_0 &= N_+ - N_- \\ L_+ &= q^{N_- - \frac{1}{2}N_0}\sqrt{q^{N_+} + q^{-N_+}}\, b_+^\dagger b_0 + b_0^\dagger b_- q^{N_+ - \frac{1}{2}N_0}\sqrt{q^{N_-} + q^{-N_-}} \\ L_- &= b_0^\dagger b_+ q^{N_- - \frac{1}{2}N_0}\sqrt{q^{N_+} + q^{-N_+}} + q^{N_+ - \frac{1}{2}N_0}\sqrt{q^{N_-} + q^{-N_-}}\, b_-^\dagger b_0 \end{aligned} \tag{2}$$

and satisfy the commutation relations

$$[L_0, L_\pm] = \pm L_\pm \qquad [L_+, L_-] = [2L_0] \tag{3}$$

where the q-numbers are defined as $[x] = (q^x - q^{-x})/(q - q^{-1})$. The Casimir operator of $so_q(3)$ can be written in the form [17]

$$\begin{aligned} C_2^{(q)} &= \tfrac{1}{2}\{L_+L_- + L_-L_+ + [2][L_0]^2\} \\ &= L_-L_+ + [L_0][L_0 + 1] = L_+L_- + [L_0][L_0 - 1]. \end{aligned} \tag{4}$$

In order to rewrite (2) in a more simplified form, we introduce the operators

$$\begin{aligned} &B_0 = q^{-\frac{1}{2}N_0} b_0 \qquad B_0^\dagger = b_0^\dagger q^{-\frac{1}{2}N_0} \\ &B_i = q^{N_i + \frac{1}{2}} b_i \sqrt{\frac{[2N_i]}{[N_i]}} \qquad B_i^\dagger = \sqrt{\frac{[2N_i]}{[N_i]}}\, b_i^\dagger q^{N_i + \frac{1}{2}} \qquad i = +, -. \end{aligned} \tag{5}$$

These operators satisfy the usual commutation relations

$$[N_i, B_i^\dagger] = B_i^\dagger \qquad [N_i, B_i] = -B_i. \tag{6}$$

One can check that in the Fock space, spanned on the normalized eigenvectors of the excitation number operators N_+, N_0, N_-, the operators (5) satisfy the relations

$$\begin{aligned} &B_0^\dagger B_0 = q^{-N_0+1}[N_0] \qquad B_0 B_0^\dagger = q^{-N_0}[N_0 + 1] \\ &B_i^\dagger B_i = q^{2N_i - 1}[2N_i] \qquad B_i B_i^\dagger = q^{2N_i+1}[2N_i + 2] \qquad i = +, - \end{aligned} \tag{7}$$

from which follow the commutation relations

$$[B_0, B_0^\dagger] = q^{-2N_0} \qquad [B_i, B_i^\dagger] = [2]q^{4N_i+1} \qquad i = +, -. \tag{8}$$

In terms of modified operators (5) the angular momentum operators (2) take the simplified form

$$\begin{aligned} L_0 &= N_+ - N_- \\ L_+ &= q^{-L_0 + \frac{1}{2}} B_+^\dagger B_0 + q^{L_0 - \frac{1}{2}} B_0^\dagger B_- \\ L_- &= q^{-L_0 - \frac{1}{2}} B_0^\dagger B_+ + q^{L_0 + \frac{1}{2}} B_-^\dagger B_0. \end{aligned} \tag{9}$$

It should be noted, however, that these expressions are not invariant with respect to the replacement $q \to q^{-1}$, which restricts us to real q.

3. $so_q(3)$-basis states

Using (9) one can check that the normalized highest-weight $so_q(3)$ state $|LL\rangle_q$, which satisfies the conditions

$$L_+|LL\rangle_q = 0 \qquad L_0|LL\rangle_q = L|LL\rangle_q \qquad \text{and} \qquad {}_q\langle LL|LL\rangle_q = 1$$

can be written in the form

$$|LL\rangle_q = q^{-\frac{1}{2}L^2} \frac{(B_+^\dagger)^L}{\sqrt{[2L]!!}}|0\rangle = \frac{(b_+^\dagger)^L}{\sqrt{[L]!}}|0\rangle. \tag{10}$$

However, these states are *not* the most general $so_q(3)$ states, since they can be multiplied by an arbitrary $so_q(3)$ scalar, which will not modify the value of L. In terms of the modified operators one can introduce the $so_q(3)$ scalars [1, 2]:

$$\begin{aligned}\tilde{S}_+ &= \frac{1}{[2]}S_+ = \frac{1}{[2]}\left\{(B_0^\dagger)^2 q^{2S_0} - B_+^\dagger B_-^\dagger q^{-2S_0}\right\}\\ \tilde{S}_0 &= S_0 = \tfrac{1}{2}\left\{N_+ + N_0 + N_- + \tfrac{3}{2}\right\} = \tfrac{1}{2}\left\{N + \tfrac{3}{2}\right\}\\ \tilde{S}_- &= \frac{1}{[2]}S_- = \frac{1}{[2]}\{q^{2S_0}(B_0)^2 - q^{-2S_0}B_+B_-\}.\end{aligned} \tag{11}$$

These operators satisfy the commutation relations

$$[\tilde{S}_0, \tilde{S}_\pm] = \pm\tilde{S}_\pm \qquad [\tilde{S}_+, \tilde{S}_-] = -[2\tilde{S}_0]_{q^2}. \tag{12}$$

From (12) it is clear that the set of $so_q(3)$ scalars $\tilde{S}_\pm, \tilde{S}_0$ close an $su_{q^2}(1,1) \sim sp_{q^2}(2,R)$ algebra. Constructing the basis, it will be simpler to use the scalars $S_\pm$, which satisfy the commutation relations

$$[S_-, S_+] = [2]^2[2S_0]_{q^2} = [2][2N+3] \qquad (S_+)^\dagger = S_-. \tag{13}$$

Therefore, the $so_q(3)$ states, characterized by an angular momentum L and its projection $M = L$, which belong to the most symmetric $[\lambda, 0, 0]$ irreducible representation of $u_q(3)$ can be written in the form

$$\left|\begin{matrix}\lambda & \\ L & L\end{matrix}\right\rangle_q = \frac{1}{N_{\lambda L}}(S_+)^{\frac{1}{2}(\lambda-L)}|LL\rangle_q \tag{14}$$

where $L = \lambda, \lambda-2, \ldots, 0$ or 1 and $|LL\rangle_q$ is a notation for the states (10). The normalization constant $N_{\lambda L}$ is determined from the condition

$${}_q\left\langle\begin{matrix}\lambda & \\ L & L\end{matrix}\right|\left.\begin{matrix}\lambda & \\ L & L\end{matrix}\right\rangle_q = 1.$$

Using the relations

$$\begin{aligned}&[S_-, (S_+)^k] = [2k](S_+)^{k-1}[2N+2k+1]\\ &S_-|LL\rangle_q = 0\\ &(S_-)^k(S_+)^k|LL\rangle_q = \frac{[2k]!![2L+2k+1]!!}{[2L+1]!!}|LL\rangle_q \qquad k = \tfrac{1}{2}(\lambda - L)\end{aligned}$$

the final result is

$$N_{\lambda L} = \sqrt{\frac{[\lambda - L]!![\lambda + L + 1]!!}{[2L+1]!!}}. \tag{15}$$

Now the states with an arbitrary projection of the momentum M can be obtained by successive application of L_- on the states (14), i.e.

$$\left|\begin{matrix}\lambda & \\ L & M\end{matrix}\right\rangle_q = \sqrt{\frac{[L+M]!}{[2L]![L-M]!}}(L_-)^{L-M}\left|\begin{matrix}\lambda & \\ L & L\end{matrix}\right\rangle_q$$

$$= \frac{q^{-\frac{1}{2}L^2}}{N_{\lambda L}}\sqrt{\frac{[L+M]!}{[2L]![L-M]!}}(S_+)^{\frac{1}{2}(\lambda-L)}(L_-)^{L-M}\frac{(B_+^\dagger)^L}{\sqrt{[2L]!!}}|0\rangle. \tag{16}$$

In order to find an explicit expression for the states (16) in form of a polynomial in terms of the operators $B_i^\dagger$ we shall make use of the auxiliary formula

$$L_-\frac{(B_+^\dagger)^x}{[2x]!!}]\frac{(B_0^\dagger)^y}{[y]!}\frac{(B_-^\dagger)^z}{[2z]!!}|0\rangle = q^{x-y-z+\frac{1}{2}}[2z+2]\frac{(B_+^\dagger)^x}{[2x]!!}\frac{(B_0^\dagger)^{y-1}}{[y-1]!}\frac{(B_-^\dagger)^{z+1}}{[2z+2]!!}|0\rangle$$

$$+q^{x+z-\frac{1}{2}}[y+1]\frac{(B_+^\dagger)^{x-1}}{[2x-2]!!}\frac{(B_0^\dagger)^{y+1}}{[y+1]!}\frac{(B_-^\dagger)^z}{[2z]!!}|0\rangle \tag{17}$$

where $x \geqslant 1$, $y \geqslant 1$ and $z \geqslant 0$. Using (17) one can prove by induction in $m \geqslant 0$ that the following relation holds

$$(L_-)^m\frac{(B_+^\dagger)^L}{[2L]!!}|0\rangle = q^{\frac{1}{2}m(2L-m)}[m]!\sum_p\frac{(B_+^\dagger)^p}{[2p]!!}\frac{(B_0^\dagger)^{2L-m-2p}}{[2L-m-2p]!}\frac{(B_-^\dagger)^{m-L+p}}{[2m-2L+2p]!!}|0\rangle \tag{18}$$

where the summation index p runs over these values, for which all exponents of the operators $B_i^\dagger$ are not negative. Replacing $m = L - M$ in (18) we obtain

$$\frac{(L_-)^{L-M}}{[L-M]!}\frac{(B_+^\dagger)^L}{[2L]!!}|0\rangle = q^{\frac{1}{2}(L^2-M^2)}\sum_{p=max(0,M)}^{\lfloor(L+M)/2\rfloor}\frac{(B_+^\dagger)^p}{[2p]!!}\frac{(B_0^\dagger)^{L+M-2p}}{[L+M-2p]!}\frac{(B_-^\dagger)^{p-M}}{[2p-2M]!!}|0\rangle. \tag{19}$$

After combining (15), (16) and (19) we obtain the following expression for the basis states

$$\left|\begin{matrix}\lambda & \\ L & M\end{matrix}\right\rangle_q = q^{-\frac{1}{2}M^2}\sqrt{\frac{[L+M]![L-M]![2L+1]}{[\lambda-L]!![\lambda+L+1]!!}}(S_+)^{\frac{1}{2}(\lambda-L)}$$

$$\times\sum_{p=max(0,M)}^{\lfloor(L+M)/2\rfloor}\frac{(B_+^\dagger)^p}{[2p]!!}\frac{(B_0^\dagger)^{L+M-2p}}{[L+M-2p]!}\frac{(B_-^\dagger)^{p-M}}{[2p-2M]!!}|0\rangle. \tag{20}$$

In order to rewrite the basis states (20) in a polynomial form, by expanding the power of S_+, one can use the q-binomial theorem [18], according to which, if the elements X and Y satisfy the condition $YX = qXY$ then

$$(X-Y)^k = \sum_{t=0}^{k}(-1)^t q^{\frac{1}{2}t(k-t)}\begin{bmatrix}k\\t\end{bmatrix}_{q^frac12}X^{k-t}Y^t. \tag{21}$$

In the present case we have

$$S_+ = \underbrace{(B_0^\dagger)^2q^{2S_0}}_{X} - \underbrace{B_+^\dagger B_-^\dagger q^{-2S_0}}_{Y} \qquad YX = q^{-4}XY.$$

Therefore, for the power of S_+ we obtain

$$(S_+)^k = \sum_{t=0}^{k}(-1)^t q^{-2t(k-t)}\begin{bmatrix}k\\t\end{bmatrix}_{q^2}\left\{(B_0^\dagger)^2q^{2S_0}\right\}^{k-t}\left\{B_+^\dagger B_-^\dagger q^{-2S_0}\right\}^t \tag{22}$$

where

$$\begin{bmatrix} k \\ t \end{bmatrix}_{q^2} = \frac{[k]_{q^2}!}{[t]_{q^2}![k-t]_{q^2}!} = \frac{[2k]!!}{[2t]!![2k-2t]!!} \qquad 2S_0 = N + \tfrac{3}{2}$$

and grouping the terms with q^N we have

$$(S_+)^k = q^{k(k+\frac{1}{2})}[2k]!! \sum_{t=0}^{k} \frac{(-1)^t q^{-(2k+1)t}}{[2t]!![2k-2t]!!} (B_+^\dagger)^t (B_0^\dagger)^{2(k-t)} (B_-^\dagger)^t q^{(k-2t)N}. \tag{23}$$

Combining (20) and (23) for $k = \frac{1}{2}(\lambda - L)$ the basis states (16) can be written in the form [1, 2, 4]

$$\left| \begin{matrix} \lambda & \\ L & M \end{matrix} \right\rangle_q = q^{\frac{1}{4}(\lambda-L)(\lambda+L+1)-\frac{1}{2}M^2} \sqrt{\frac{[L+M]![L-M]![\lambda-L]!![2L+1]}{[\lambda+L+1]!!}}$$
$$\times \sum_{t=0}^{(\lambda-L)/2} \sum_{p=max(0,M)}^{\lfloor (L+M)/2 \rfloor} \frac{(-1)^t q^{-(\lambda+L+1)t}}{[2t]!![\lambda-L-2t]!!} \frac{(B_+^\dagger)^{p+t}}{[2p]!!}$$
$$\times \frac{(B_0^\dagger)^{\lambda+M-2p-2t}}{[L+M-2p]!} \frac{(B_-^\dagger)^{p+t-M}}{[2p-2M]!!} |0\rangle. \tag{24}$$

4. Vector operators

The $so_q(3)$ tensor operators must satisfy the commutation relations, which directly follow from the expression for the adjoint action of the corresponding algebra [16–18]. By definition, the irreducible tensor operator T_m^j of rank j according to $so_q(3)$ satisfies the commutation relations

$$[L_0, T_m^j] = m\ T_m^j$$
$$[L_\pm, T_m^j]_{q^m} q^{L_0} = \sqrt{[j \mp m][j \pm m + 1]}\ T_{m\pm 1}^j. \tag{25}$$

The generalization of the Wigner–Eckart theorem to the case of the algebra $so_q(3)$ is

$$\langle \alpha', L'M' | T_m^j | \alpha, LM \rangle = (-1)^{2j} \frac{{}_qC_{LM,jm}^{L'M'}}{\sqrt{[2L'+1]}} \langle \alpha', L' \| T^j \| \alpha, L \rangle \tag{26}$$

where $|\alpha, LM\rangle$ are orthonormalized basis vectors of the irreducible representation ${}_qD^L$ of the algebra $so_q(3)$ and ${}_qC_{L_1M_1,L_2M_2}^{LM}$ are the Clebsch–Gordan coefficients [17, 18] of the same algebra. It should be noted that the operator

$$R_m^j = (-1)^m q^{-m} (T_{-m}^j)^\dagger \tag{27}$$

where the superscript † denotes Hermitian conjugation, transforms in the same way (25) as the tensor operator T_m^j, i.e. it also is an irreducible $so_q(3)$ tensor operator of rank j.

In order to construct irreducible $so_q(3)$ vector operators $T_m^\dagger$ and $\tilde{T}_m$ we start from the observation

$$[L_0, B_+^\dagger] = B_+^\dagger \tag{28}$$

and suppose that the highest-weight component of the vector operator $T_m^\dagger$ is

$$T_{+1}^\dagger = \omega B_+^\dagger q^{\alpha N_+ + \beta N_0 + \gamma N_- + \delta} \tag{29}$$

where $\alpha, \beta, \gamma, \delta$ and ω are real constants to be determined. As irreducible first-rank $so_q(3)$ tensor operator, $T_m^\dagger$ $(m = 0, \pm 1)$ must satisfy the relations

$$[L_0, T_m^\dagger] = m\, T_m^\dagger \qquad [L_\pm, T_m^\dagger]_{q^m} q^{L_0} = \sqrt{[1 \mp m][2 \pm m]}\, T_{m\pm 1}^\dagger. \tag{30}$$

The same relations hold for the operators

$$\tilde{T}_m = (-1)^m q^{-m} (T_{-m}^\dagger)^\dagger = (-1)^m q^{-m} T_{-m} \tag{31}$$

where $(T_m^\dagger)^\dagger = T_m$ and † denotes Hermitian conjugation. According to (30), the condition

$$[L_+, T_{+1}^\dagger]_q = 0 \tag{32}$$

is satisfied, if $\alpha + 2 = \beta = \gamma$ for any real constants ω and δ, and the operator $T_{+1}^\dagger$ can be written as

$$T_{+1}^\dagger = \omega B_+^\dagger q^{-2N_+ + \beta N + \delta}. \tag{33}$$

Further by the action of L_- we get all other components of $T_m^\dagger$

$$T_0^\dagger = \omega \sqrt{[2]} B_0^\dagger q^{-2N_+ + \beta N + \delta + \frac{1}{2}} \qquad T_{-1}^\dagger = \omega \{ B_-^\dagger q^{2N_+ + (\beta - 2)N + \delta} - (q - q^{-1}) B_+ (B_0^\dagger)^2 q^{-2N_+ + \beta N + \delta + 2} \}. \tag{34}$$

One can check that the condition

$$[L_-, T_{-1}^\dagger]_{q^{-1}} = 0 \tag{35}$$

holds for any values of the parameters β, δ and ω. From these expressions it is clear that $T_m^\dagger$ $(m = 0, \pm 1)$ is a vector operator according to $so_q(3)$. The components of the corresponding conjugated vector operator $\tilde{T}_m$ $(m = 0, \pm 1)$ given by (31) are

$$\begin{aligned} \tilde{T}_{+1} &= -\omega \{ q^{2N_+ + (\beta - 2)N + \delta - 1} B_- - (q - q^{-1}) q^{-2N_+ + \beta N + \delta + 1} B_+^\dagger (B_0)^2 \} \\ \tilde{T}_0 &= \omega \sqrt{[2]} q^{-2N_+ + \beta N + \delta + \frac{1}{2}} B_0 \\ \tilde{T}_{-1} &= -\omega q^{-2N_+ + \beta N + \delta + 1} B_+. \end{aligned} \tag{36}$$

Using the vector operators $T_m^\dagger$ and $\tilde{T}_m$ one can construct the coupled operators [6, 17]

$$A_M^L = [T^\dagger \otimes \tilde{T}]_M^L = \sum_{m,n} {}_{q^{-1}}C_{1m,1n}^{LM} T_m^\dagger \tilde{T}_n \qquad L = 0, 1, 2. \tag{37}$$

Actually we use a particular case of a product of two irreducible tensor operators acting on a single vector [17]. If $T_m^\dagger$ and $\tilde{T}_m$ are vector operators according to $so_q(3)$ then the operators (37) are irreducible tensors of rank $L = 0, 1, 2$ according to the same algebra. Their Hermitian conjugates are

$$(A_M^L)^\dagger = (-1)^M q^{-M} A_{-M}^L. \tag{38}$$

In order to determine the parameters β, δ and ω we shall take into account that from the generators L_+, L_0, L_- of the algebra $so_q(3)$ one can construct a first-rank tensor J^1 [17, 18] according to this algebra as

$$J_{\pm 1}^1 = \mp \frac{1}{\sqrt{[2]}} q^{-L_0} L_\pm \tag{39}$$

$$\begin{aligned} J_0^1 &= \frac{1}{[2]} \{ q L_+ L_- - q^{-1} L_- L_+ \} = \frac{1}{[2]} \{ q [2L_0] + (q - q^{-1}) L_- L_+ \} \\ &= \frac{1}{[2]} \left\{ q[2L_0] + (q - q^{-1}) \left(C_2^{(q)} - [L_0][L_0 + 1] \right) \right\} \end{aligned} \tag{40}$$

where $C_2^{(q)}$ is the second-order Casimir operator (4) of $so_q(3)$. After imposing the condition

$$J^1_M = -\sqrt{\frac{[4]}{[2]}} A^1_M \qquad M = 0, \pm 1 \tag{41}$$

where A^1_M is a first-rank tensor (37) and J^1_M is also a first-rank tensor (39) we obtain

$$\omega = \frac{1}{\sqrt{[2]}} \qquad \beta = 1 \qquad \delta = -\tfrac{1}{2} \qquad tqs\alpha + 2 = \beta = \gamma. \tag{42}$$

The final expressions for the components of the vector operator $T^\dagger_m$ are

$$\begin{aligned} T^\dagger_{+1} &= \frac{1}{\sqrt{[2]}} B^\dagger_+ q^{-2N_+ + N - \frac{1}{2}} \\ T^\dagger_0 &= B^\dagger_0 q^{-2N_+ + N} \\ T^\dagger_{-1} &= \frac{1}{\sqrt{[2]}} \left\{ B^\dagger_- q^{2N_+ - N - \frac{1}{2}} - (q - q^{-1}) B_+ (B^\dagger_0)^2 q^{-2N_+ + N + \frac{3}{2}} \right\}. \end{aligned} \tag{43}$$

The corresponding expressions for $\tilde{T}_m$ can be obtained from (31)

$$\begin{aligned} \tilde{T}_{+1} &= -\frac{1}{\sqrt{[2]}} \left\{ q^{2N_+ - N - \frac{3}{2}} B_- - (q - q^{-1}) q^{-2N_+ + N + \frac{1}{2}} B^\dagger_+ (B_0)^2 \right\} \\ \tilde{T}_0 &= q^{-2N_+ + N} B_0 \\ \tilde{T}_{-1} &= -\frac{1}{\sqrt{[2]}} q^{-2N_+ + N + \frac{1}{2}} B_+. \end{aligned} \tag{44}$$

In terms of $T^\dagger_m$ and $\tilde{T}_m$ one can construct the scalars

$$\begin{aligned} R_+ &= -\sqrt{[3]} [T^\dagger \otimes T^\dagger]^0_0 = -\sqrt{[3]} \sum_{m,n} {}_{q^{-1}}C^{00}_{1m,1n} T^\dagger_m T^\dagger_n \\ R_- &= -\sqrt{[3]} [\tilde{T} \otimes \tilde{T}]^0_0 = -\sqrt{[3]} \sum_{m,n} {}_{q^{-1}}C^{00}_{1m,1n} \tilde{T}_m \tilde{T}_n \end{aligned} \tag{45}$$

and can easily check that

$$R_+ = S_+ q^{2S_0} \qquad R_- = q^{2S_0} S_- \qquad (R_+)^\dagger = R_- \tag{46}$$

where S_+, S_0, S_- are given by (11).

Now using the relations

$$\begin{aligned} &[T^\dagger_{+1}, T^\dagger_0]_{q^2} = 0 \\ &[T_{+1}, T^\dagger_{+1}]_{q^{-2}} = q^{2N} \qquad \text{or} \qquad [\tilde{T}_{-1}, T^\dagger_{+1}]_{q^{-2}} = -q^{2N+1} \end{aligned}$$

which follow from the explicit form (43), (44) of these operators, one can compute by successive application of L_- the commutation relations between $\tilde{T}_m$ and $T^\dagger_m$

$$\begin{aligned} &[T^\dagger_{+1}, T^\dagger_0]_{q^2} = 0 \qquad [\tilde{T}_0, \tilde{T}_{-1}]_{q^2} = 0 \\ &[T^\dagger_0, T^\dagger_{-1}]_{q^2} = 0 \qquad [\tilde{T}_{+1}, \tilde{T}_0]_{q^2} = 0 \\ &[T^\dagger_{+1}, T^\dagger_{-1}] = (q - q^{-1})(T^\dagger_0)^2 \qquad [\tilde{T}_{+1}, \tilde{T}_{-1}] = (q - q^{-1})(\tilde{T}_0)^2 \end{aligned} \tag{47}$$

and

$$
\begin{aligned}
&[\tilde{T}_0, T_{+1}^\dagger] = 0 \qquad [\tilde{T}_{-1}, T_0^\dagger] = 0 \\
&[\tilde{T}_{+1}, T_{+1}^\dagger]_{q^2} = 0 \qquad [\tilde{T}_{-1}, T_{-1}^\dagger]_{q^2} = 0 && (48a)\\
&[\tilde{T}_{+1}, T_0^\dagger] = (q^2 - q^{-2})T_{+1}^\dagger \tilde{T}_0 \qquad [\tilde{T}_0, T_{-1}^\dagger] = (q^2 - q^{-2})T_0^\dagger \tilde{T}_{-1} \\
&[\tilde{T}_{-1}, T_{+1}^\dagger]_{q^{-2}} = -q^{2N+1} \\
&[\tilde{T}_0, T_0^\dagger] = q^{2N} + q^{-1}(q^2 - q^{-2})T_{+1}^\dagger \tilde{T}_{-1} && (48b)\\
&[\tilde{T}_{+1}, T_{-1}^\dagger]_{q^{-2}} = -q^{2N-1} + q^{-1}(q^2 - q^{-2}) \left\{ T_0^\dagger \tilde{T}_0 + (q - q^{-1})T_{+1}^\dagger \tilde{T}_{-1} \right\}
\end{aligned}
$$

which are similar to the results obtained by Quesne [6].

5. Matrix elements of the quadrupole operator

The aim of this section is the calculation of the reduced matrix elements of the q-deformed quadrupole operator Q^2, namely, the quantities

$$\langle \lambda, L + 2 \| Q^2 \| \lambda, L \rangle \qquad \text{and} \qquad \langle \lambda, L \| Q^2 \| \lambda, L \rangle$$

where the operator Q^2 is

$$Q_M^2 = \sqrt{\frac{[3][4]}{[2]}} A_M^2 \qquad A_M^2 = [T^\dagger \otimes \tilde{T}]_M^2 = \sum_{m,n} q^{-1} C_{1m,1n}^{2M} T_m^\dagger \tilde{T}_n. \tag{49}$$

In (49) the factor has been chosen to agree with the usual convention in the classical case when $q \to 1$. Other reduced matrix elements do not occur, since in the most symmetric representation of $u_q(3)$ only states with equal parity of λ and L exist. From the tensor structure of the operator A^2 for its zero component we have

$$A_0^2 \left| \begin{matrix} \lambda & \\ L & L \end{matrix} \right\rangle_q = \boldsymbol{a} \left| \begin{matrix} \lambda & \\ L+2 & L \end{matrix} \right\rangle_q + \boldsymbol{b} \left| \begin{matrix} \lambda & \\ L & L \end{matrix} \right\rangle_q \tag{50}$$

and it is clear that the coefficients $\boldsymbol{a}, \boldsymbol{b}$ determine the reduced matrix elements of the tensor A^2.

The highest-weight vector of the basis states (14) can be expressed in terms of vector operators (43) as follows

$$\left| \begin{matrix} \lambda & \\ L & L \end{matrix} \right\rangle_q \frac{(S_+)^k}{N_{\lambda L}} \frac{(T_{+1}^\dagger)^L}{\sqrt{[L]_{q^2}!}} |0\rangle \tag{51}$$

where $k = \frac{1}{2}(\lambda - L)$ and the normalization constant $N_{\lambda L}$ is determined in (15). Therefore,

$$
\begin{aligned}
A_0^2 \left| \begin{matrix} \lambda & \\ L & L \end{matrix} \right\rangle_q &= A_0^2 \frac{(S_+)^k}{N_{\lambda L}} \frac{(T_{+1}^\dagger)^L}{\sqrt{[L]_{q^2}!}} |0\rangle \\
&= \frac{1}{N_{\lambda L}\sqrt{[L]_{q^2}!}} \left\{ (S_+)^k A_0^2 + [A_0^2, (S_+)^k] \right\} (T_{+1}^\dagger)^L |0\rangle. && (52)
\end{aligned}
$$

Now, in order to calculate the action of A_0^2 on the highest-weight vector (51), we use the identities

$$\begin{aligned}
[A_2^2, (S_+)^k] &= q^{2k-2}[2k](S_+)^{k-1}(T_{+1}^\dagger)^2 q^{2S_0} \\
[A_1^2, (S_+)^k] &= \sqrt{\frac{[4]}{[2]}} q^{2k-1}[2k](S_+)^{k-1} T_0^\dagger T_{+1}^\dagger q^{2S_0} \\
[A_0^2, (S_+)^k] &= \sqrt{\frac{[4]}{[3][2]}} q^{2k}[2k](S_+)^{k-1}\left\{S_+ q^{2S_0+1} + [3]T_{-1}^\dagger T_{+1}^\dagger\right\} q^{2S_0}
\end{aligned} \tag{53}$$

which can be obtained from (47), (48*a*), (48*b*) and successive application of the operator L_-. Finally using the relations

$$A_0^2 (T_{+1}^\dagger)^L|0\rangle = -\sqrt{\frac{[2]}{[3][4]}}\frac{q^3[2L]}{[2]}(T_{+1}^\dagger)^L|0\rangle \tag{54}$$

$$T_{-1}^\dagger (T_{+1}^\dagger)^{L+1}|0\rangle = \frac{q^{-2L-2}}{[2L+4][2L+3]}(L_-)^2(T_{+1}^\dagger)^{L+2}|0\rangle - \frac{q^{L+\frac{5}{2}}[2L+2]}{[2][2L+3]} S_+(T_{+1}^\dagger)^L|0\rangle \tag{55}$$

we obtain the expressions for the coefficients $\boldsymbol{a}, \boldsymbol{b}$ in the expansion (50)

$$\boldsymbol{a} = \frac{q^{\lambda-2L-\frac{1}{2}}}{[2L+3]}\sqrt{\frac{[3][4]}{[2]}}\sqrt{\frac{[\lambda-L][\lambda+L+3][2L+2]}{[2][2L+5]}} \tag{56}$$

and

$$\boldsymbol{b} = -\frac{q^{\lambda+\frac{5}{2}}[2L]}{[2][2L+3]}\sqrt{\frac{[2]}{[3][4]}}\{q^{L-\frac{1}{2}}[\lambda-L] + q^{-L+\frac{1}{2}}[\lambda+L+3]\}. \tag{57}$$

From (56), (57) and the Wigner–Eckart theorem (26) one immediately obtains the reduced matrix elements of the quadrupole operator Q^2 defined by (49)

$$\langle\lambda, L+2\|Q^2\|\lambda, L\rangle = \frac{q^{\lambda-\frac{1}{2}}}{[2]}\sqrt{\frac{[3][4]}{[2]}}\sqrt{\frac{[\lambda-L][\lambda+L+3][2L+4][2L+2]}{[2L+3]}} \tag{58}$$

$$\langle\lambda, L\|Q^2\|\lambda, L\rangle = -\frac{q^{\lambda-\frac{1}{2}}}{[2]}\sqrt{\frac{[2L][2L+1][2L+2]}{[2L-1][2L+3]}}\{q^{L-\frac{1}{2}}[\lambda-L] + q^{-L+\frac{1}{2}}[\lambda+L+3]\} \tag{59}$$

which is in agreement with the classical case when $q \to 1$. Taking into account the Wigner–Eckart theorem and the symmetry properties of the q-deformed Clebsch–Gordan coefficients it can be shown also that the reduced matrix element (58) of the q-deformed quadrupole operator (49) has the following symmetry property

$$\langle\lambda, L+2\|Q^2\|\lambda, L\rangle = \langle\lambda, L\|Q^2\|\lambda, L+2\rangle. \tag{60}$$

For small values of the deformation parameter τ ($q = e^\tau$, τ-real) the reduced matrix elements (58), (59) can be represented in Taylor expansions

$$\begin{aligned}
\langle\lambda, L+2\|Q^2\|\lambda, L\rangle &= \sqrt{\frac{6(\lambda-L)(\lambda+L+3)(L+2)(L+1)}{2L+3}} \\
&\quad \times\{1 + (\lambda-\tfrac{1}{2})\tau + (\tfrac{2}{3}\lambda^2 + \tfrac{1}{2}L^2 + \tfrac{3}{2}L + \tfrac{65}{24})\tau^2 + O(\tau^3)\}
\end{aligned} \tag{61}$$

$$\langle\lambda, L\|Q^2\|\lambda, L\rangle = -(2\lambda+3)\sqrt{\frac{L(L+1)(2L+1)}{(2L-1)(2L+3)}}\left\{1+\frac{2}{2\lambda+3}\{\lambda(\lambda+1)-L(L+1)\}\tau\right.$$
$$\left.+\frac{1}{3(2\lambda+3)}\{(2\lambda+15)L(L+1)+(2\lambda+1)(2\lambda^2+2\lambda+3)\}\tau^2+\mathrm{O}(\tau^3)\right\} \tag{62}$$

and if $\tau = 0$ one obtains the classical expressions for the corresponding reduced matrix elements.

6. Conclusion

In the present paper we gave an approach for the construction of irreducible tensor operators in the case of the q-deformed chain $u_q(3) \supset so_q(3)$ for the most symmetric representations $[\lambda, 0, 0]$ of the $u_q(3)$ algebra. In this way we have calculated the reduced matrix elements (58), (59) of the q-deformed quadrupole operator (49).

Of great interest from a physical point of view are the $E2$-transition probabilities ($B[E2]$-factors), which in the classical case are expressed by means of the reduced matrix elements of the $u(3)$-quadrupole operator. It can be shown that the $B[E2]$-factors corresponding to the chain $u_q(3) \supset so_q(3)$ are of the form

$$B[E2; (\lambda, L+2) \to (\lambda, L)]_q = \frac{1}{[2L+5]}|\langle\lambda, L+2\|Q^2\|\lambda, L\rangle|^2 \tag{63}$$

where Q^2 is the q-deformed quadrupole operator (49). Likewise the reduced matrix element (59) is related to the deformation of the physical system in the state with angular momentum L.

It should be noted that the results obtained here, strictly speaking, are valid only for real values of the deformation parameter q. On the other hand the comparison of the experimental data with the predictions of a number of physical models [19, 20], based on the q-deformed $su_q(2)$ algebra, shows that one can achieve a good agreement between theory and experiment only if q is a pure phase ($q = \mathrm{e}^{\mathrm{i}\tau}$). Nevertheless, we suppose, however, that these quadrupole operators describe some q-deformed excitations (q-deformed phonons) and the obtained results are a necessary step to further investigations.

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